

Nonlinear evolution wave equation for an artery with an aneurysm: an exact solution obtained by the modified method of simplest equation

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Abstract

We study propagation of traveling waves in a blood filled elastic artery with an axially symmetric dilatation (an idealized aneurysm) in long-wave approximation. The processes in the injured artery are modelled by equations for the motion of the wall of the artery and by equation for the motion of the fluid (the blood). For the case when balance of nonlinearity, dispersion and dissipation in such a medium holds the model equations are reduced to a version of the Korteweg-deVries-Burgers equation with variable coefficients. Exact travelling-wave solution of this equation is obtained by the modified method of simplest equation where the differential equation of Riccati is used as a simplest equation. Effects of the dilatation geometry on the travelling-wave profile are considered.

1 Introduction

Theoretical and experimental investigation of pulse wave propagation in human arteries has a long history. Recently the research on blood flow has intensified because the fast development of methods for studying nonlinear

systems [1]–[17] allowed the researchers to achieve great progress in understanding nonlinear waves in fluid-solid structures, such as blood-filled human arteries. Over the past decade, however, the scientific efforts have been concentrated on theoretical investigations of nonlinear wave propagation in arteries with a variable radius through the blood. Clearing how local imperfections appeared in an artery can disturb the blood flow can help in predicting the nature and main features of various cardiovascular diseases, such as stenoses and aneurysms. In order to examine propagation of nonlinear waves in a stenosed artery, Tay and co-authors treated the artery as a homogeneous, isotropic and thin-walled elastic tube with an axially symmetric stenosis. The blood was modeled as an incompressible inviscid fluid [18], Newtonian fluid with constant viscosity [19], and Newtonian fluid with variable viscosity [20]. Using a specific perturbation method, in a long-wave approximation the authors obtained the forced Korteweg-de Vries (KdV) equation with variable coefficients [18], forced perturbed KdV equation with variable coefficients [19], and forced Korteweg-de Vries–Burgers (KdVB) equation with variable coefficients as evolution equations [20]. The same theoretical frame was used in [21], [22] to examine nonlinear wave propagation in an artery with a variable radius. Considering the artery as a long inhomogeneous prestretched thin elastic tube with an imperfection (presented at large by an unspecified function $f(z)$), and the blood as an incompressible inviscid fluid the authors reached again the forced KdV equation with variable coefficients. Apart from solitary propagation waves in such a system, in [22], possibility of periodic waves was discussed at appropriate initial conditions. In the present work we shall focus on consideration of the blood flow through an artery with a local dilatation (an aneurysm). The aneurysm is a localized, blood-filled balloon-like bulge in the wall of a blood vessel [23]. In many cases, its rupture causes massive bleeding with associated high mortality. Motivated by investigations in [18]–[22], the main goal of this paper is to investigate effects of the aneurismal geometry and the blood characteristics on the propagation of nonlinear waves through an injured artery. For that purpose, we use a reductive perturbation method to obtain the nonlinear evolution equation. Exact solution of this equation is obtained by using the modified method of simplest equation. Recently, this method has been widely used to obtain general and particular solutions of economic, biological and physical models, represented by partial differential equations. The paper is organized as follows. A brief description about the derivation of equations governing the blood flow through a dilated artery is presented in Sec. 2. In Sec. 3 we derive a basic evolution equation in long-wave approximation. A travelling wave solution of this equation is obtained in Sec. 4. Numerical simulations of the solution are presented in Sec. 5. The main conclusions based on the obtained

results are summarized in Sec. 6 of the paper.

2 Mathematical Formulation of the Basic Model

In order to derive the model of a blood-filled artery with an aneurysm we have to consider two types equations which represent (i) the motion of the arterial wall and (ii) the motion of the blood. To model such a medium we shall treat the artery as a thin-walled incompressible prestretched hyperelastic tube with a localized axially symmetric dilatation. We shall assume the blood to be an incompressible viscous fluid. A brief formulation of the above-mentioned equations follows in the next two subsections.

2.1 Equation of the wall

It is well-known, that for a healthy human, the systolic pressure is about 120 mm Hg and the diastolic pressure is 80 mm Hg. Thus, the arteries are initially subjected to a mean pressure, which is about 100 mm Hg. Moreover, the elastic arteries are initially prestretched in an axial direction. This feature minimizes its axial deformations during the pressure cycle. For example, experimental studies declare that the longitudinal motion of arteries is very small [24], and it is due mainly to strong vascular tethering and partly to the predominantly circumferential orientation of the elastin and collagen fibers. Taking into account these observations, and following the methodology applied in [18]–[21], we consider the artery as a circularly cylindrical tube with radius R_0 , assuming that such a tube is subjected to an initial axial stretch λ_z and a uniform inner pressure $P_0^*(Z)$. Under the action of such a variable pressure the position vector of a generic point on the tube can be described by

$$\mathbf{r}_0 = [r_0 + f^*(z^*)]\mathbf{e}_r + z^*\mathbf{e}_z, \quad z^* = \lambda_z Z^* \quad (1)$$

where \mathbf{e}_r and \mathbf{e}_z are the unit basic vectors in the cylindrical polar coordinates, r_0 is the deformed radius at the origin of the coordinate system, Z^* is axial coordinate before the deformation, z^* is the axial coordinate after static deformation and $f^*(z^*)$ is a function describing the dilatation geometry. We shall specify the concrete form of $f^*(z^*)$ later. Upon the initial static deformation, we shall superimpose only a dynamical radial displacement $u^*(z^*, t^*)$, neglecting the contribution of axial displacement because of the experimental observations, given above. Then, the position vector r of a generic point on the tube can be expressed by

$$\mathbf{r} = [r_0 + f^*(z^*) + u^*]\mathbf{e}_r + z^*\mathbf{e}_z \quad (2)$$

The arc-lengths along meridional and circumferential curves respectively, are:

$$ds_z = [1 + (f^{*'} + \frac{\partial u^*}{\partial z^*})^2]^{1/2} dz^*, \quad ds_\theta = [r_0 + f^* + u^*] d\theta \quad (3)$$

In this way, the stretch ratios in the longitudinal and circumferential directions in final configuration are presented by

$$\lambda_1 = \lambda_z \Lambda, \quad \lambda_2 = \frac{1}{R_0} (r_0 + f^* + u^*) \quad (4)$$

where

$$\Lambda = [1 + (f^{*'} + \frac{\partial u^*}{\partial z^*})^2]^{1/2} \quad (5)$$

The notation ' $'$ ' denotes the differentiation of f^* with respect to z^* . Then, the unit tangent vector t along the deformed meridional curve and the unit exterior normal vector n to the deformed tube are expressed by

$$\mathbf{t} = \frac{(f^{*'} + \frac{\partial u^*}{\partial z^*}) \mathbf{e}_r + \mathbf{e}_z}{\Lambda}, \quad \mathbf{n} = \frac{\mathbf{e}_r - (f^{*'} + \frac{\partial u^*}{\partial z^*}) \mathbf{e}_z}{\Lambda} \quad (6)$$

According to the assumption made about material incompressibility the following restriction holds:

$$h = \frac{H}{\lambda_1 \lambda_2} \quad (7)$$

where H and h are the wall thicknesses before and after deformation, respectively. In addition, for hyperelastic materials, the tensions in longitudinal and circumferential directions take the form:

$$T_1 = \frac{\mu^* H}{\lambda_2} \frac{\partial \Pi}{\partial \lambda_1}, \quad T_2 = \frac{\mu^* H}{\lambda_1} \frac{\partial \Pi}{\partial \lambda_2} \quad (8)$$

where $\mu^* \Pi$ is the strain energy density function of wall material as μ^* is the material shear modulus. Although the elastic properties of an injured wall section differ from those of the healthy part, here, we assume that the wall is homogeneous, i.e. μ^* is a constant through the axis z . Finally, according to the second Newton's law, the equation of radial motion of a small tube element placed between the planes $z^* = \text{const}$, $z^* + dz^* = \text{const}$, $\theta = \text{const}$ and $\theta + d\theta = \text{const}$ obtains the form:

$$-\frac{\mu^*}{\lambda_z} \frac{\partial \Pi}{\partial \lambda_2} + \mu^* R_0 \frac{\partial}{\partial z^*} \left\{ \frac{(f^{*'} + \partial u^* / \partial z^*)}{\Lambda} \frac{\partial \Pi}{\partial \lambda_1} \right\} + \frac{P_r^*}{H} (r_0 + f^* + u^*) \Lambda = \rho_0 \frac{R_0}{\lambda_z} \frac{\partial^2 u^*}{\partial t^{*2}} \quad (9)$$

where P_r^* is the fluid reaction force, which shall be specified later, and ρ_0 is the mass density of the tube material.

2.2 Equation of the fluid

Experimental studies over many years demonstrated that blood behaves as an incompressible non-Newtonian fluid because it consists of a suspension of cell formed elements in a liquid well-known as blood plasma. However, in the larger arteries (with a vessel radius larger than 1 mm) it is plausible to assume that the blood has an approximately constant viscosity, because the vessel diameters are essentially larger than the individual cell diameters. Thus, in such vessels the non-Newtonian behavior becomes insignificant and the blood can be considered as a Newtonian fluid. Here, for our convenience we assume a ‘hydraulic approximation’ and apply an averaging procedure with respect to the cross-sectional area to the Navier–Stokes equations. Then, we obtain

$$\frac{\partial A^*}{\partial t^*} + \frac{\partial}{\partial z^*}(A^* \omega^*) = 0 \quad (10)$$

$$\frac{\partial \omega^*}{\partial t^*} + \omega^* \frac{\partial \omega^*}{\partial z^*} + \frac{1}{\rho_f} \frac{\partial P^*}{\partial z^*} = \frac{\mu_f}{\rho_f} \frac{\partial^2 \omega^*}{\partial z^{*2}} + \frac{2\mu_f}{r_f^2 \rho_f} \left(r \frac{\partial V_z^*}{\partial r} \right) \Big|_{r=r_f} \quad (11)$$

where A^* denotes the inner cross-sectional area, i.e., $A^* = \pi r_f^2$ as $r_f = r_0^* + f^* + u^*$ is the final radius of the tube after deformation. Substituting A^* into Eq (10) leads to

$$2 \frac{\partial u^*}{\partial t^*} + 2\omega^* [f^{*'} + \frac{\partial u^*}{\partial z^*}] + [r_0 + f^*(z^*) + u^*] \frac{\partial \omega^*}{\partial z^*} = 0 \quad (12)$$

We introduce the following non-dimensional quantities

$$t^* = \left(\frac{R_0}{c_0}\right)t, \quad z^* = R_0 z, \quad u^* = R_0 u, \quad f^* = R_0 f, \quad \omega^* = c_0 \omega, \quad \mu_f = c_0 R_0 \rho_f \nu,$$

$$P^* = \rho_f c_0^2 p, \quad r_0 = R_0 \lambda_\theta, \quad c_0^2 = \frac{\mu^* H}{\rho_f R_0}, \quad m = \frac{\rho_0 H}{\rho_f R_0}, \quad V_z^* = c_0 V_z, \quad r = R_0 x$$

and replace them into Eqs (12), (11) and (9), respectively. Then, the final model takes the form:

$$2 \frac{\partial u}{\partial t} + 2\omega [f' + \frac{\partial u}{\partial z}] + [\lambda_\theta + f(z) + u] \frac{\partial \omega}{\partial z} = 0 \quad (13)$$

$$\frac{\partial \omega}{\partial t} + \omega \frac{\partial \omega}{\partial z} + \frac{\partial p}{\partial z} = \nu \frac{\partial^2 \omega}{\partial z^2} + \frac{2\nu}{(\lambda_\theta + f + u)^2} \left(\frac{\partial V_z}{\partial x} \right) \Big|_{x=\lambda_\theta+f+u} \quad (14)$$

$$p = \frac{m}{\lambda_z(\lambda_\theta + f(z) + u)} \frac{\partial^2 u}{\partial t^2} + \frac{1}{\lambda_z(\lambda_\theta + f(z) + u)} \frac{\partial \Pi}{\partial \lambda_2} - \frac{1}{(\lambda_\theta + f(z) + u)} \frac{\partial}{\partial z} \left(\frac{f' + \partial u / \partial z}{\Lambda} \right) \frac{\partial \Pi}{\partial \lambda_1} + \nu \frac{(f' + \partial u / \partial z) \omega}{\lambda_\theta + f + u} \quad (15)$$

where λ_θ is the initial stretch ratio in a circumferential direction.

3 Derivation of the evolution equation in a long-wave approximation

In this section we shall use the long-wave approximation to employ the reductive perturbation method [25] for studying the propagation of small-but-finite amplitude waves in a fluid-solid structure system, presented by Eqs (13)–(15). In the long-wave limit, it is assumed that the variation of radius along the axial coordinate is small compared with the wave length. Because this condition is valid for large arteries, next we shall treat the problem as boundary value problem. We linearize Eqs (13)–(15), and search for their harmonic kind solution. Next, according to the obtained dispersion equation it will be appropriate to introduce the following type of stretched coordinates

$$\xi = \epsilon^{1/2}(z - ct), \quad \tau = \epsilon^{3/2}z \quad (16)$$

where ϵ is a small parameter measuring the weakness of nonlinearity, dispersion and dissipation, and c is a scale parameter which will be determined from the solution. Then, $z = \epsilon^{-3/2}\tau$, and $f(\epsilon^{-3/2}\tau) = \epsilon\chi(\xi, \tau)$. Thus, the variables u, ω and p are functions of the variables (ξ, τ) and the small parameter ϵ . Taking into account the effect of dilatation, we assume f to be of order of $5/2$, i.e.

$$\chi(\xi, \tau) = \epsilon h(\tau) \quad (17)$$

In addition, taking into account the effect of viscosity, the order of viscosity is assumed to be $O(1/2)$, i.e.

$$\nu = \epsilon^{1/2}\bar{\nu} \quad (18)$$

The last assumption ensures balance of nonlinearity, dispersion and dissipation in the system. In order to eliminate the unknown term $(\frac{\partial V_z}{\partial x})$ in Eq (14), we use the transformation [26]

$$y = (\lambda_\theta + \epsilon h(\tau) + u - x)\epsilon^2 \quad (19)$$

For the long wave limit, it is assumed that the field quantities may be expanded into asymptotic series as

$$\begin{aligned} u &= \epsilon u_1 + \epsilon^2 u_2 + \dots, \quad \omega = \epsilon \omega_1 + \epsilon^2 \omega_2 + \dots, \\ p &= p_0 + \epsilon p_1 + \epsilon^2 p_2 + \dots, \quad \lambda_1 \cong \lambda_z, \\ \lambda_2 &= \lambda_\theta + \epsilon(u_1 + h) + \epsilon^2(u_2 + (u_1 + h)^2) + \dots, \quad \frac{1}{\lambda_\theta \lambda_z} \frac{\partial \Pi}{\partial \lambda_z} = \gamma_0 \\ \frac{1}{\lambda_\theta \lambda_z} \frac{\partial \Pi}{\partial \lambda_\theta} &= \beta_0 + \beta_1(u_1 + h)\epsilon + (\beta_1 u_2 + \beta_2(u_1 + h)^2)\epsilon^2 + \dots \end{aligned} \quad (20)$$

where

$$\beta_0 = \frac{1}{\lambda_\theta \lambda_z} \frac{\partial \Pi}{\partial \lambda_\theta}, \quad \beta_1 = \frac{1}{\lambda_\theta \lambda_z} \frac{\partial^2 \Pi}{\partial \lambda_\theta^2}, \quad \beta_2 = \frac{1}{2\lambda_\theta \lambda_z} \frac{\partial^3 \Pi}{\partial \lambda_\theta^3} \quad (21)$$

Substituting (16)–(20) into Eqs (13)–(15), we obtain the following differential sets:

$O(\epsilon)$ equations

$$-2c \frac{\partial u_1}{\partial \xi} + \lambda_\theta \frac{\partial \omega_1}{\partial \xi} = 0, \quad -c \frac{\partial \omega_1}{\partial \xi} + \frac{\partial p_1}{\partial \xi} = 0, \quad p_1 = \gamma_1(u_1 + h) \quad (22)$$

$O(\epsilon^2)$ equations

$$\begin{aligned} -2c \frac{\partial u_2}{\partial \xi} + 2\omega_1 \frac{\partial u_1}{\partial \xi} + \lambda_\theta \frac{\partial \omega_2}{\partial \xi} + [u_1 + h] \frac{\partial \omega_1}{\partial \xi} + \lambda_\theta \frac{\partial \omega_1}{\partial \tau} &= 0 \\ -c \frac{\partial \omega_2}{\partial \xi} + \omega_1 \frac{\partial \omega_1}{\partial \xi} + \frac{\partial p_2}{\partial \xi} + \frac{\partial p_1}{\partial \tau} - \overline{\nu} \frac{\partial^2 \omega_1}{\partial \xi^2} &= 0 \\ p_2 = \left(\frac{mc^2}{\lambda_\theta \lambda_z} - \gamma_0 \right) \frac{\partial^2 u_1}{\partial \xi^2} + \gamma_1 u_2 + \gamma_2 (u_1 + h)^2 & \end{aligned} \quad (23)$$

From Eqs (22), we obtain

$$u_1 = U(\xi, \tau), \quad \omega_1 = \frac{2c}{\lambda_\theta} U, \quad p_1 = \frac{2c^2}{\lambda_\theta} U + \gamma_1 h \quad (24)$$

as $\gamma_1 = \frac{2c^2}{\lambda_\theta}$. We introduce (24) into (23), and obtain

$$-2c \frac{\partial u_2}{\partial \xi} + \frac{4c}{\lambda_\theta} U \frac{\partial U}{\partial \xi} + \lambda_\theta \frac{\partial \omega_2}{\partial \xi} + 2c \frac{\partial U}{\partial \tau} + \frac{2c}{\lambda_\theta} (U + h) \frac{\partial U}{\partial \xi} = 0 \quad (25)$$

$$-c \frac{\partial \omega_2}{\partial \xi} + \frac{4c^2}{\lambda_\theta^2} U \frac{\partial U}{\partial \xi} + \frac{2c^2}{\lambda_\theta} \frac{\partial U}{\partial \tau} + \gamma_1 h' + \frac{\partial p_2}{\partial \xi} - \frac{4c^2}{\lambda_\theta^2} \frac{\partial^2 U}{\partial \xi^2} = 0 \quad (26)$$

$$p_2 = \left(\frac{mc^2}{\lambda_\theta \lambda_z} - \gamma_0 \right) \frac{\partial^2 U}{\partial \xi^2} + \gamma_1 u_2 + \gamma_2 U^2 + \gamma_2 h(\tau) U + \gamma_2 h(\tau)^2 \quad (27)$$

Replacing Eq (27) into Eq (26), and eliminating ω_2 between Eq (25) and Eq (26), the final evolution equation takes the form:

$$\frac{\partial U}{\partial \tau} + \mu_1 U \frac{\partial U}{\partial \xi} - \mu_2 \frac{\partial^2 U}{\partial \xi^2} + \mu_3 \frac{\partial^3 U}{\partial \xi^3} + \mu_4(\tau) \frac{\partial U}{\partial \xi} + \mu(\tau) = 0 \quad (28)$$

where

$$\begin{aligned}\mu_1 &= \frac{5}{2\lambda_\theta} + \frac{\gamma_2}{\gamma_1}, \quad \mu_2 = \frac{\bar{\nu}}{\lambda_\theta}, \quad \mu_3 = \frac{m}{4\lambda_z} - \frac{\gamma_0}{2\gamma_1}, \\ \mu_4(\tau) &= h(\tau)\left(\frac{1}{2\lambda_\theta} + \frac{\gamma_2}{\gamma_1}\right), \quad \mu(\tau) = \frac{1}{2}h'(\tau)\end{aligned}\tag{29}$$

and

$$\gamma_1 = \beta_1 - \frac{\beta_0}{\lambda_\theta}, \quad \gamma_2 = \beta_2 - \frac{\beta_1}{\lambda_\theta}\tag{30}$$

Finally we have to objectify the idealized aneurysm shape. For an idealized abdominal aortic aneurysm (AAA), $h(\tau) = \delta \exp(\frac{-\tau^2}{2L^2})$, where δ is the aneurysm height, i.e. $\delta = r_{max} - r_0$, and L is the aneurysm length [45]. In order to normalize these geometric quantities, we non-dimensionalize δ by the inlet radius (diameter). Then, the non-dimensional coefficient can be presented by $\delta' = DI - 1$, where $DI = 2r_{max}/2r_0 = D_{max}/D_0$ is a geometric measure of AAA, which is known as a diameter index or a dilatation index [46]. In the same manner, the aneurysm length L is normalized by the maximum aneurysm diameter (D_{max}), i.e. $l' = L/D_{max} = 1/SI$, where SI is a ratio, which is known as a sacular index of AAA [46]. For AAAs, D_{max} varies from 3 cm to 8.5 cm, and L varies from 5 cm to 10–12 cm.

4 Analytical solution for the nonlinear evolution equation: Application of the modified method of simplest equation

In this section we shall derive a progressive wave solution for the variable coefficients evolution equation, presented by (28). We shall make change of the function and the variables in the the evolution equation with variable coefficients as follows:

Let us introduce $U(\xi, \tau) = V(\xi, \tau) + \phi(\tau)$. Replacing this substitution into (28), leads to:

$$\frac{\partial V}{\partial \tau} + \mu_1(V - \int \mu(\tau)d\tau)\frac{\partial V}{\partial \xi} - \mu_2\frac{\partial^2 V}{\partial \xi^2} + \mu_3\frac{\partial^3 U}{\partial \xi^3} + \mu_4(\tau)\frac{\partial V}{\partial \xi} = 0.$$

Now, we introduce the coordinate transformation

$$\tau' = \tau, \quad \xi' = \xi - \int [\mu_4(\tau) + \mu_1 \int \mu(\tau)d\tau]d\tau$$

Then, Eq (28) is reduced to the generalized KdVB equation:

$$\frac{\partial V}{\partial \tau'} + \mu_1 \frac{\partial V}{\partial \xi'} - \mu_2 \frac{\partial^2 V}{\partial \xi'^2} + \mu_3 \frac{\partial^3 V}{\partial \xi'^3} = 0. \quad (31)$$

Next, we shall find an analytical solution of Eq (31) applying the modified method of simplest equation [27]–[36]. This method has its roots in the research of Kudryashov and ther authors [37]–[42]. The short description of the modified method of simplest equation is as follows. First of all by means of an appropriate ansatz (for an example the traveling-wave ansatz) the solved of nonlinear partial differential equation for the unknown function η is reduced to a nonlinear ordinary differential equation that includes η and its derivatives with respect to the traveling wave coordinate ζ

$$\Phi(\eta, \eta_\zeta, \eta_{\zeta\zeta}, \dots) = 0 \quad (32)$$

Then the finite-series solution

$$\eta(\zeta) = \sum_{\mu=-\kappa}^{\kappa_1} a_\mu [g(\zeta)]^\mu \quad (33)$$

is substituted in (32). a_μ are coefficients and $g(\zeta)$ is solution of simpler ordinary differential equation called simplest equation. Let the result of this substitution be a polynomial of $g(\zeta)$. Eq. (33) is a solution of Eq.(32) if all coefficients of the obtained polynomial of $g(\zeta)$ are equal to 0. This condition leads to a system of nonlinear algebraic equations. Each nontrivial solution of the last system leads to a solution of the studied nonlinear partial differential equation. In addition, in order to obtain the solution of Eq.(32) by the above method we have to ensure that each coefficient of the obtained polynomial of $g(\zeta)$ contains at least two terms. To do this within the scope of the modified method of the simplest equation we have to balance the highest powers of $g(\zeta)$ that are obtained from the different terms of the solved equation of kind (32). As a result of this we obtain an additional equation between some of the parameters of the equation and the solution. This equation is called a balance equation.

Introducing the transformation $\zeta = \xi' - v\tau'$, we search for solution of (31) of kind $V = V(\zeta) = \sum_{r=0}^q a_r g_\zeta^r$, where $g'_\zeta = \sum_{j=0}^m b_j g^j$. Here a_r and b_j are parameters, and $g(\zeta)$ is a solution of some ordinary differential equation, referred to as the simplest equation. The balance equation is $q = 2m - 2$. We assume that $m = 2$, i.e. the equation of Riccati will play the role of simplest equation. Then

$$V = a_0 + a_1 g + a_2 g^2, \quad \frac{dg}{d\zeta} = b_0 + b_1 g + b_2 g^2 \quad (34)$$

The differential equation of Riccati can be written as

$$\left(\frac{dg}{d\zeta}\right)^2 = c_0 + c_1g + c_2g^2 + c_3g^3 + c_4g^4 \quad (35)$$

where

$$c_0 = b_0^2; \quad c_1 = 2b_0b_1; \quad c_2 = 2b_0b_2 + b_1^2; \quad c_3 = 2b_1b_2; \quad c_4 = b_2^2 \quad (36)$$

and its solutions are given in [29]. The relationships among the coefficients of the solution and the coefficients of the model are derived by solving a system of five algebraic equations, and they are

$$\begin{aligned} a_0 &= -\frac{1}{25} \frac{-3\mu_2^2 - 30\mu_2\mu_3b_1 + 75\mu_3^2b_1^2 + 25v\mu_3}{\mu_1\mu_3}; \\ a_1 &= -\frac{12}{5} \frac{b_2(5\mu_3b_1 - \mu_2)}{\mu_1}; \quad a_2 = -12 \frac{\mu_3b_2^2}{\mu_1}; \quad b_0 = \frac{1}{100} \frac{25\mu_3^2b_1^2 - \mu_2^2}{b_2\mu_3^2} \end{aligned} \quad (37)$$

Here b_1, b_2 are free parameters. Then, the solution of the evolution equation with constant coefficients (Eq (31)) is

$$\begin{aligned} V(\zeta) &= -\frac{1}{25} \frac{-3\mu_2^2 - 30\mu_2\mu_3b_1 + 75\mu_3^2b_1^2 + 25v\mu_3}{\mu_1\mu_3} - \\ &\quad -\frac{12}{5} \frac{b_2(5\mu_3b_1 - \mu_2)}{\mu_1} g(\zeta) - 12 \frac{\mu_3b_2^2}{\mu_1} g(\zeta)^2 \end{aligned} \quad (38)$$

where

$$\begin{aligned} g(\zeta) &= -\frac{b_1}{2b_2} - \frac{\Delta}{2b_2} \tanh\left(\frac{\Delta(\zeta + \zeta_0)}{2}\right) + \\ &\quad + \frac{\exp\left(\frac{\Delta(\zeta + \zeta_0)}{2}\right)}{2 \cosh\left(\frac{\Delta(\zeta + \zeta_0)}{2}\right) \frac{b_2}{\Delta} + 2C^* \exp\left(\frac{\Delta(\zeta + \zeta_0)}{2}\right) \cosh\left(\frac{\Delta(\zeta + \zeta_0)}{2}\right)} \end{aligned} \quad (39)$$

In Eq. (39) $\Delta = \sqrt{b_1^2 - 4b_0b_2} > 0$, and ζ_0 and C^* are constants of integration. The solution of the evolution equation with variable coefficients (Eq (28)) is

$$U(\xi, \tau) = V(\zeta) - \int \mu(\tau) d\tau \quad (40)$$

where

$$\zeta = \xi - v\tau - \int [\mu_1 \int \mu(\tau) d\tau + \mu_4(\tau)] d\tau \quad (41)$$

5 Numerical findings and discussions

In order to see the effect of dilatation on the wave profiles of investigated quantities, we need the values of coefficients $\beta_0, \beta_1, \beta_2, \gamma_0, \gamma_1, \gamma_2, \mu_1, \mu_2, \mu_4(\tau)$ and $\mu(\tau)$. For that purpose, the constitutive relation for tube material must be specified. Here, unlike [18]–[22], we assume that the arterial wall is an incompressible, anisotropic and hyperelastic material. The mechanical behaviour of such a material can be defined by the strain energy function of Fung for arteries [43]:

$$\Pi = C(e^Q - 1), \quad Q = C_1 E_{QQ}^2 + C_2 E_{ZZ}^2 + 2C_3 E_{QQ} E_{ZZ} \quad (42)$$

where E_{QQ} and E_{ZZ} are the Green–Lagrange strains in the circumferential and axial directions, respectively, and C, C_1, C_2, C_3 are material constants. Taking into account that $E_{QQ} = 1/2(\lambda_\theta^2 - 1)$ and $E_{ZZ} = 1/2(\lambda_z^2 - 1)$, we substitute (42) in (21) and (30), and obtain:

$$\begin{aligned} \beta_0 &= \frac{1}{\lambda_z} \left(\frac{C_1}{2} + C_3(\lambda_z^2 - 1) \right) F(\lambda_\theta \lambda_z) \\ \beta_1 &= \frac{1}{\lambda_z \lambda_\theta} \left(\frac{C_1}{2} + C_3(\lambda_z^2 - 1) \right) (1 + \lambda_\theta^2 \left(\frac{C_1}{2} + C_3(\lambda_z^2 - 1) \right)) F(\lambda_\theta \lambda_z) \\ \beta_2 &= \frac{1}{2\lambda_z} \left(\frac{C_1}{2} + C_3(\lambda_z^2 - 1) \right)^2 (3 + \lambda_\theta^2 \left(\frac{C_1}{2} + C_3(\lambda_z^2 - 1) \right)) F(\lambda_\theta \lambda_z) \\ \gamma_0 &= \frac{1}{\lambda_\theta} \left(\frac{C_2}{2} + C_3(\lambda_\theta^2 - 1) \right) F(\lambda_\theta \lambda_z), \quad \gamma_1 = \frac{1}{\lambda_z} \left(\frac{C_1}{2} + C_3(\lambda_z^2 - 1) \right)^2 F(\lambda_\theta \lambda_z), \\ \gamma_2 &= \frac{1}{\lambda_z} \left(\frac{C_1}{2} + C_3(\lambda_z^2 - 1) \right) \left(\frac{\lambda_\theta^2}{2} \left(\frac{C_1}{2} + C_3(\lambda_z^2 - 1) \right)^2 + \frac{5}{2} \left(\frac{C_1}{2} + C_3(\lambda_z^2 - 1) \right) \right. \\ &\quad \left. - \frac{1}{\lambda_\theta^2} \right) F(\lambda_\theta \lambda_z) \end{aligned} \quad (43)$$

where

$$F(\lambda_\theta \lambda_z) = C \exp \left(\frac{C_1}{4} (\lambda_\theta^2 - 1) + \frac{C_2}{4} (\lambda_z^2 - 1) + \frac{C_3}{2} (\lambda_\theta^2 - 1) (\lambda_z^2 - 1) \right) \quad (44)$$

The numerical values of material coefficients in (44) are as follows: $C = 2.5, C_1 = 14.5, C_2 = 7, C_3 = 0.1$. They were derived in [44] from experimental data of human aortic wall segments applying a specific inverse technique. Assuming the initial deformation $\lambda_z = 1.5, \lambda_\theta = 1.2$, we obtain the following values for the coefficients: $\beta_0 = 554.97, \beta_1 = 5374, \beta_2 = 27872.89, \gamma_0 = 333.36, \gamma_1 = 4911.52, \gamma_2 = 23394.55$. Then, the numerical values of the

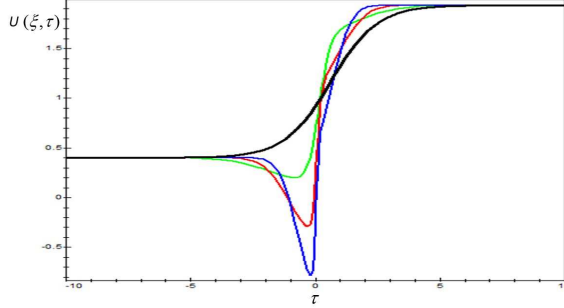


Figure 1: Wave profiles of the radial displacement U at $D_{max}=2$ cm (the black line); $D_{max}=3$ cm (the green line); $D_{max}=5$ cm (the red line); $D_{max}=7$ cm (the blue line) ($L = 5cm$)

coefficients in Eq. (28) are:

$$\begin{aligned} \mu_1 &= 6.85; \mu_2 = 0.42; \mu_3 = -0.017; \\ \mu_4(\tau) &= 5.36\delta' \exp(-\tau^2/2l'^2),; \mu(\tau) = -\delta'\tau \exp(-\tau^2/2l'^2)/2l'^2. \end{aligned} \quad (45)$$

Using these numerical values, the travelling-wave solution of Eq.(28) for various time ξ ($\xi=0..5$ (years)) is plotted in Fig (1). It is seen that in absence of arterial dilatation (at $D_{max} = D_0$) the wave demonstrates a pure kink (the black line in the figure). In the case of an aneurismal artery, however, a slight wave drop, followed by a prompt wave jump is observed (see the other lines in the figure). In addition, the wave drop region diminishes when the value of the maximal aneurysm diameter (D_{max}) decreases. This result seems to be admissible from the point of view of arterial mechanics.

6 Conclusions

Modelling the injured artery as a thin-walled prestretched, anisotropic and hyperelastic tube with a local imperfection (an aneurysm), and the blood as a Newtonian fluid we have derived an evolution equation for propagation of nonlinear waves in this complex medium. Numerical values of the model parameters are determined for concrete mechanical characteristics of the arterial wall and concrete aneurismal geometry. We have obtained an analytical solution of the evolution equation of a travelling-wave type. The numerical simulations of this solution demonstrate that for a healthy artery the wave is of a pure kink type, while wave drop and jump effects are observed when a local arterial dilatation arises.

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